Linking Caplets and Swaptions Prices in the LMM-SABR Model

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Abstract

We use (and improve upon) a recent time-homogeneous extension of the SABR-LMM aproach described in Rebonato (2007) to develop a quick and accurate analytical approximation to the implied swaption prices given the forward-rate SABR-LMM parameters. This approximation can be used for studies of calibration of a forward-rate-based LMM-SABR model to (portions of) the swaption matrix, to determine the CMS drift corrections (see Hagan (2003)) and to study the congruence between the caplet and swaption markets.

1 Introduction and Motivation

The SABR 'model' has become an industry standard for communicating, interpolating and extrapolating the prices of plain-vanilla caplets and swaptions. (For a description of the SABR model see, e.g., Hagan et al (2002)). The LIBOR Market Model (LMM) and its stochastic-volatility extensions have become similarly common to price exotics. The deterministic-volatility LMM is described, e.g., in Brace, Gatarek and Musiela (1996) and Jamshidian (1997). Some of the many stochastic-volatility extensions can be found, e.g., in Andersen and Andreasen (1998), Joshi and Rebonato (2003), Piterbarg (2003), Piterbarg (2005), Rebonato and Joshi (2004), Rebonato and Kainth (2004), White and Rebonato (2007).

Labordere (2006) presents a possible unification of the LMM (BGM) and the SABR models using concepts borrowed from hyperbolic geometry. He finds relationships between forward-rate and swap-rate volatilities that are exact at the zeroth order. He proves that approaches such as the one presented in this paper (based on the 'freezing' of suitably chosen stochastic quantities) cannot be exactly correct away from the at-the-money strike. The relevant question is therefore whether the approximations that we introduce below are acceptable in the range of parameters normally found in typical pricing applications.

The SABR model and the LMM do not directly 'talk to each other'. Ultimately, the SABR approach provides a series of independent 'snapshots' of the caplet smiles (one snapshot for each maturity), but does not link these still frames into a coherent 'movie' (i.e., does not prescribe well-defined joint dynamics). To overcome this shortcoming, Rebonato (2007) recently introduced an extension of the forward-rate-based LMM that enjoys the following features:

- it is constructed to be the smallest possible perturbation of a perfectly-time-homogenous model that recovers the market prices (to a certain accuracy);
- it is of almost immediate calibration;
- with the same parameters it recovers the prices of caplets of all strikes and maturities (out to 30 years) generated by the SABR snapshots within approximately half a vega.

Since the approach is relatively new, it is briefly reviewed in Section 2; there we also introduce the notation and present an improvement on the approximations in Rebonato (2007).

In this work we assume that the forward rates follow LMM-SABR dynamics whose parameters are available from the procedure described in Rebonato (2007). Under this assumption we present a semi-analytic approximation to the SABR parameters for swaptions implied by the LMM-SABR dynamics for the forward rates. More precisely, the parameters of the LMM-SABR approach (plus correlation assumptions, see later) imply a set of prices for swaptions. One could laboriously obtain the swaption prices using a Monte Carlo simulation and fit the SABR model to these swaption prices. We provide a useful analytical shortcut that allows us to 'guess' very accurately the swaption SABR parameters that would obtain from this fitting without having to carry out any Monte Carlo simulation. We also examine the sensitivity of the implied swaption prices to the inputs of the forward-rate-based LMM-SABR model. Such an approximation would be extremely useful for calibrating the forward-rate-based LMM-SABR model to (a sub-set of) the swaption matrix and in order to obtain the drift correction for CMS products in a forward-rate-based LMM approach.

2 Review and Improvement of the Forward-Rate LMM-SABR Approach

In the deterministic-volatility (no-smile) LMM the desirable feature of time homogeneity of the caplet implied volatility surface can only be achieved if the instantaneous volatilities of the forward rates, $\sigma_i(t, T_i)$, are purely a function, say, $g()$, of the residual time to maturity, $T_i - t$, of the associated forward rate:

$$\sigma_i(t, T_i) = g(T_i - t) \tag{1}$$

(see, e.g., Brigo and Mercurio (2001) or Rebonato (2002)). For practical applications one wants a simple and analytical form for the function $g()$. A common choice for $g(T_i - t)$ is:

$$\sigma_i(\tau_i) = (a + b\tau_i) \exp(-c\tau_i) + d \tag{2}$$

with $\tau_i = T_i - t$. See, e.g., Brigo and Mercurio (2001), Rebonato (2002, 2005), White and Rebonato (2007) for a justification of the choice and description of the properties of this function.

With a function $g()$ parametrized by a small number of coefficients, perfect recovery of an exogenous set of at-the-money prices cannot in general be recovered even in the absence of smiles. In the deterministic-volatility setting a common approach to overcome this problem is to 'correct' the time homogeneous function as little as possible by introducing expiry-specific factors, k_i, chosen so as to ensure that

$$\widehat{\sigma}_i T_i^2 = k_i^2 \int_0^{T_i} g(T_i - t)^2 dt \tag{3}$$

where $\widehat{\sigma}_i$ indicates the 'implied' (root-mean-squared) volatility of the ith caplet. If the function $g()$ is parametrized by a set of coefficients $\{\alpha_k, k = 1, 2, ..., m\}$, then a good fitting procedure is to impose that the parameters should be chosen so as to ensure that the correction factors k_i are as close to unity for all forward rates. When this constraint is imposed, it is generally found that instantaneous-volatility functions for forward rates naturally display the humped shaped that has become part of market lore. See Dodds (1998) for an empirical study, and Rebonato (2005) and White and Rebonato (2007) for indirect evidence from swaption prices. We would like to retain as much as possible of this intuition when moving to the stochastic-volatility setting.

Consider the following SABR dynamics under the terminal measure, Q^T, of forward rate f_t^T:

$$df_t^T = \sigma_t^T \left(f_t^T\right)^{\beta_{SABR}^T} dz_t^T \tag{4}$$

$$\frac{d\sigma_t^T}{\sigma_t^T} = v^T dw_t^T \tag{5}$$

$$E\left[dz_t^T dw_t^T\right] = \phi_{SABR}^T dt \tag{6}$$

The SABR dynamics is therefore fully described by the initial conditions, f_0^T, σ_0^T, and by the expiry-dependent parameters $\beta_{SABR}^T, \phi_{SABR}^T, v^T$.[1]

As for the LMM specification, we obtain our results under the same terminal measure, Q^T, under which the forward rate is driftless. The SABR parameters above are assumed to be available from a previous SABR fitting to market caplet prices for all maturities. They implicitly determine the caplet prices for

[1] To lighten notation, we often omit the dependence of the correlation ϕ and of the exponent β_{SABR} on the expiry T.

all strikes. We want to determine the parameters of a LMM such that the LMM caplet prices for all the same strikes and maturities are as close as possible to the SABR caplet prices. We want to do so in a financially desirable way. To this effect, consider the dynamics of the forward rate of maturity T under the same measure, $\mathbb{Q}^T$, in the LMM:

$$df_t^i = \left(f_t^i\right)^{\beta_i} s_t^{T_i} dz_t^i \tag{7}$$

$$s_t^{T_i} = k_t^{T_i} g_t^{T_i} \tag{8}$$

$$dk_t^{T_i} = \mu_k^i dt + k_t^{T_i} h_t^{T_i} dw_t^i \tag{9}$$

$$E[dz_t^i dz_t^j] = \rho_{ij} dt \tag{10}$$

$$E[dw_t^i dw_t^j] = \theta_{ij} dt \tag{11}$$

$$E[dw_t^i dz_t^j] = \phi_{ij} dt \tag{12}$$

with $g_t^T = g(T-t)$. Note that if $k_t^{T_i} \equiv s_t^{T_i} / g_t^{T_i}$ were a deterministic function of t and T_i (or a constant) we would be in the traditional deterministic-volatility LMM setting. As for the drift term, μ_k^i, for an arbitrary numeraire in general this term is non-zero for all the forward-rate volatilities. It stems from imposing the condition that the relative price, given by the forward rate times its natural payoff divided by the chosen numeraire, should be a martingale. See, eg, Rebonato (2004). In particular, see Labordere (2006, 2007) for the drift term when the swap annuity is chosen as numeraire. We note in passing that we experimented with a variety of numeraires, and always found this term to be very small, and of negligible numerical impact. We therefore follow Labordere (2007) and set the drift term μ_k^i to zero in the approximate formulae that we derive below.

To retain time-homogeneity as much as possible, we impose that the volatility of volatility should have the functional form

$$h(t,T) = h(T-t) \tag{13}$$

with the function $h()$ parametrized by a set of parameters $[l, m, n, ...]$. Therefore

$$k_t^T = k_0^T \exp\left[\int_0^t \left\{-\frac{1}{2}h^2(T-s)ds + h(T-s)dw_s\right\}\right] \tag{14}$$

and

$$s_t^T = g_t^T k_t^T = g_t^T k_0^T \exp\left[\int_0^t \left\{-\frac{1}{2}h^2(T-s)ds + h(T-s)dw_s\right\}\right] \tag{15}$$

As far as caplets are concerned, the calibration problem is how to choose the parameters of the two functions g and h. In a deterministic-volatility setting the quantities $\{k^T\}$ would be fully determined by the requirement that each caplet should be perfectly priced. See Equation (3). This is no longer the case in the stochastic-volatility setting. We therefore heuristically impose that the parameters a, b, c and d should be chosen in such a way as to match as closely as possible the expectation at time 0 of σ_s^T, σ_0^T. More precisely, we minimize over a, b, c and d the sum, χ^2, of the squared discrepancies:

$$\chi^2 = \sum_i^N \left[\sigma_0^{T_i} - \widehat{g}(T_i)\right]^2 \tag{16}$$

where the sum over i runs over the N caplet expiries and

$$\widehat{g}(T_i) = \sqrt{\frac{1}{T_i}\int_0^{T_i} \left[(a + b\tau_i)\exp(-c\tau_i) + d\right]^2 d\tau_i} \tag{17}$$

The initial values $k_0^{T_i}$ in Equation (3) are then chosen so as to provide exact recovery of the quantities $\sigma_0^{T_i}$:

$$\sigma_0^{T_i} = k_0^{T_i}\widehat{g}(T_i) \tag{18}$$

If the chosen function $g(\tau)$ allows for a good fit to the dependence of the initial SABR value $\sigma_0^{T_i}$ on the maturities T_i, these correction factors will all be close to 1.

We now move to the second function, $h_t^T = h(T - t)$, that describes the volatility of volatility. We assume that the SABR maturity-dependent volatility-of-volatility, v^{T_i}, coefficients are available from a previous market fit. In the original paper (Rebonato (2007)), it was recommended to use a function of the form:

$$h_\tau = (\alpha + \beta\tau)\exp(-\gamma\tau) + \delta \tag{19}$$

with $\tau = T - t$. The parameters α, β, γ and δ are chosen to minimize the sum, χ^2, of the squared discrepancies:

$$\chi^2 = \sum_i^N \left[v^{T_i} - \widehat{h}(T_i)\right]^2 \tag{20}$$

where, again, $\widehat{h}(T_i)$ denotes the root-mean-squared of the function h to time T_i. Small correction factors, ξ_i, were then applied to ensure the exact recovery of the root-mean-squared volatility of volatility, $\widehat{h}(T_i)$:

$$v^{T_i} = \xi_i\widehat{h}(T_i) \tag{21}$$

Here we propose a modification of this procedure that significantly improves its accuracy. The rationale for the new approximation is as follows.

In the limit as the volatility of volatility goes to zero, we can clearly guarantee that the distributions of f_{SABR} and f_{LMM} match at option expiry (and hence that the European option prices match) by imposing

$$(\sigma_0)^2 T = k_0 \int_0^T g(t)^2 dt \tag{22}$$

When the volatility is stochastic, however, what matters for the terminal distribution of the forward rates (and hence for the pricing of caplets) is neither the terminal value of the volatility (variance) nor the average value of the volatility of volatility over the life of the European option. To see this, consider the two time-dependent volatility of volatility functions, $h_A(t)$ and $h_B(t)$ depicted in Fig 1 against the same function $g(t)$. They have been constructed to have the same root-mean-squared volatility, $\widehat{h}(T_i)$. Despite this, it is easy to see that it does matter for caplet pricing when the high volatility of volatility period occurs: to begin with, if, for instance, the volatility of volatility were high when the volatility itself were low (curve labelled $h(B)$ in Fig 1), this stochasticity of the volatility would effectively be 'wasted'. More importantly, even if the function $g()$ were flat, *when* the volatility of volatility is concentrated has a large effect on the smile: if it were concentrated towards the expiry, for instance, it would again have a small impact on option prices. We try to obtain an approximation that reflects these insights.

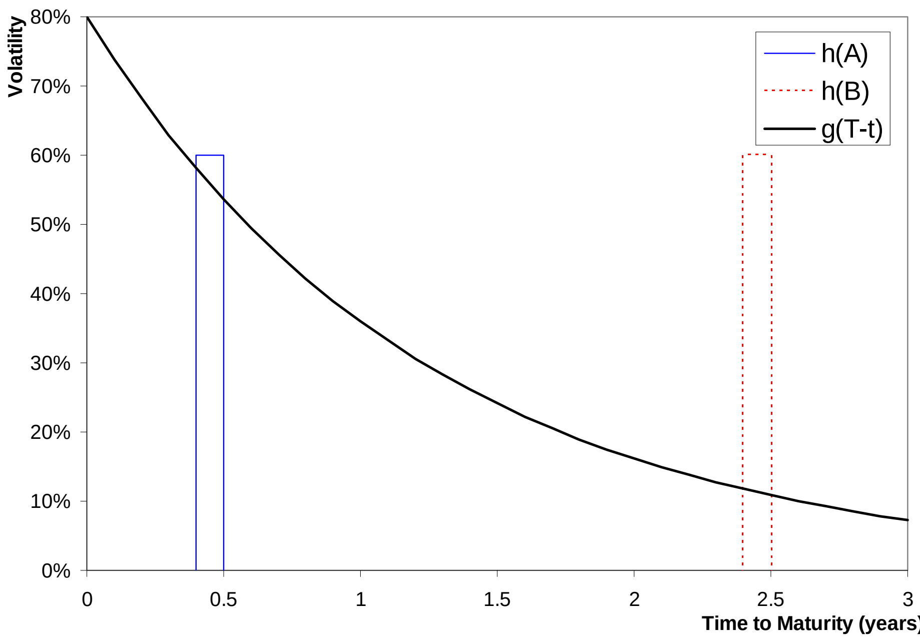

Figure 1: Two possible volatility-of-volatility functions, $h_A(t)$ and $h_B(t)$, against a time-varying volatility function, $g(t)$. See the text for an explanation.

The price of a European option conditional on a particular realisation of the path of the stochastic volatility is

$$(\text{Call}|\text{vol-path}) = p(0,T)\mathbb{E}\left[(f_T - k)^+|\text{vol-path}\right] \tag{23}$$

Then the tower rule[2] gives

$$\text{Call} = p(0,T)\mathbb{E}\left[\mathbb{E}\left[(f_T - k)^+|\text{vol-path}\right]\right] \tag{24}$$

We define the integral of the stochastic volatility squared as

$$\Gamma_T = \int_0^T (\sigma_t)^2 \, dt \tag{25}$$

which is itself a stochastic quantity; we define a particular realisation [3] of this integral as γ_T. If the correlation in the SABR model, ϕ, were zero, we could replace the vol-path by Γ_T. We therefore have

$$\begin{aligned} \text{Call} &= p(0,T)\mathbb{E}\left[\mathbb{E}\left[(f_T - k)^+\Big|\Gamma_T = \gamma_T\right]\right] \\ &= p(0,T)\int_{-\infty}^{\infty} \mathbb{E}\left[(f_T - k)^+\Big|\Gamma_T = \gamma_T\right] f(\gamma_T)d\gamma_T \end{aligned} \tag{26}$$

where $f(\gamma_T)$ is the unknown density of Γ_T. We now proceed in a heuristic fashion and approximate this unknown density by a Dirac delta at its expectation, and obtain

$$\begin{aligned} \text{Call} &\approx p(0,T)\int_{-\infty}^{\infty} \mathbb{E}\left[(f_T - k)^+\Big|\Gamma_T = \gamma_T\right] \delta(\gamma_T - \mathbb{E}[\Gamma_T])d\gamma_T \\ &= p(0,T)\mathbb{E}\left[(f_T - k)^+\Big|\Gamma_T = \mathbb{E}[\Gamma_T]\right] \\ &= p(0,T)\mathbb{E}\left[(f_T - k)^+\Big|\Gamma_T = \mathbb{E}\left[\int_0^T \sigma_t^2 dt\right]\right] \end{aligned} \tag{27}$$

The equivalent expression for the LMM is

$$\text{Call} \approx p(0,T)\mathbb{E}\left[(f_T - k)^+\Big|\Gamma_T = \mathbb{E}\left[\int_0^T g(t)^2 k(t)^2 dt\right]\right] \tag{28}$$

We do not claim that either of these approximations is accurate, but we try to make the same error when dealing with the SABR and the LMM dynamics. Looking at equations (27) and (28), it is clear that in order to match the call prices we must impose the condition

$$\begin{aligned} \mathbb{E}\left[\int_0^T \sigma_t^2 dt\right] = \mathbb{E}\left[\int_0^T g(t)^2 k(t)^2 dt\right] &\Longrightarrow \\ \int_0^T \mathbb{E}[\sigma_t^2] dt = \int_0^T g(t)^2 \mathbb{E}[k(t)^2] dt &\Longrightarrow \\ \left(\frac{\sigma_0}{\nu}\right)^2 \left(e^{\nu^2 T} - 1\right) = (k_0)^2 \int_0^T g(t)^2 e^{t\widehat{h}_t^2} dt & \end{aligned} \tag{29}$$

[2] Also known as the law of total expectation, or the law of iterated expectations.

[3] Here we follow the usual notation of using a capital letter to denote a random variable/distribution, and the lower case letter to denote a particular draw from that distribution.

where, as above, $\widehat{h_t} = \sqrt{\frac{1}{t}\int_0^t (h_s)^2 ds}$ denotes the RMS value of $h()$. Taylor expanding both sides of this expression to second order gives

$$\begin{aligned}&(\sigma_0)^2 T + \frac{(\sigma_0)^2 \nu^2 T^2}{2} + \frac{(\sigma_0)^2 \nu^4 T^3}{3!} + \ldots \\ &= (k_0)^2 \int_0^T g(t)^2 dt + (k_0)^2 \int_0^T g(t)^2 \widehat{h_t}^2 t dt + \frac{(k_0)^2}{2} \int_0^T g(t)^2 \widehat{h_t}^4 t^2 dt + \ldots\end{aligned} \tag{30}$$

Equating the first term gives

$$(\sigma_0)^2 T = (k_0)^2 \int_0^T g(t)^2 dt$$

This is the same condition we had obtained before (see Equation (22)). The second term then gives:

$$\frac{(\sigma_0)^2 \nu^2 T^2}{2} = (k_0)^2 \int_0^T g(t)^2 \widehat{h_t}^2 t dt \Longrightarrow \tag{31}$$

$$\nu = \frac{k_0}{\sigma_0 T} \left(2 \int_0^T g(t)^2 \widehat{h_t}^2 t dt \right)^{1/2} \tag{32}$$

If the first two terms are matched, and the functions $g()$ and $h()$ are not too pathological, then one can hope that further terms will not be too dissimilar - this is the reason for expanding *both* sides of equation (29). Note that, as hoped for, the approximation that we have obtained depends on *when* the volatility and the volatility of volatility are large or small.

For the above to hold, it is necessary that $\rho_{\text{SABR}} \approx \rho_{\text{LMM}}$ and $\beta_{\text{SABR}} \approx \beta_{\text{LMM}}$; so finally, the plausible choices are made:

$$\phi_{\text{SABR}} = \phi_{\text{LMM}} \tag{33}$$

$$\beta_{\text{SABR}} = \beta_{\text{LMM}} \tag{34}$$

Figs 2 and 3 display the very high quality of the approximation. The curves in these two figures show:

- the implied volatilities obtained by inputting the SABR parameters into the Hagan (2003) approximate formula - curve labelled 'Market(Hagan)';
- the implied volatilities obtained by inputting the SABR parameters into a Monte Carlo simulation of the true SABR process - curve labelled 'Market (MC)'; this is our benchmark;
- the implied volatilities obtained using the approximation presented in Rebonato (2007) - curve labelled SV-LMM (old);

- the implied volatilities obtained using the approximation presented suggested above (Equation (32)) - curve labelled SV-LMM (new); this is the curve to be most directly compared with the benchmark.

With the new approximation the largest error for the five-year (ten-year) caplet is of 11 (9) basis points in volatility for a strike of 3.66% (3.67%) when the forward is at 5.19% (5.60%), *five to ten times smaller than what was obtained with the previous approximation.* It is important to stress that these figures show that the accuracy of the approximation does not depend on the correlation between the forward rate and its volatility being close to zero, but on a cancellation of errors.

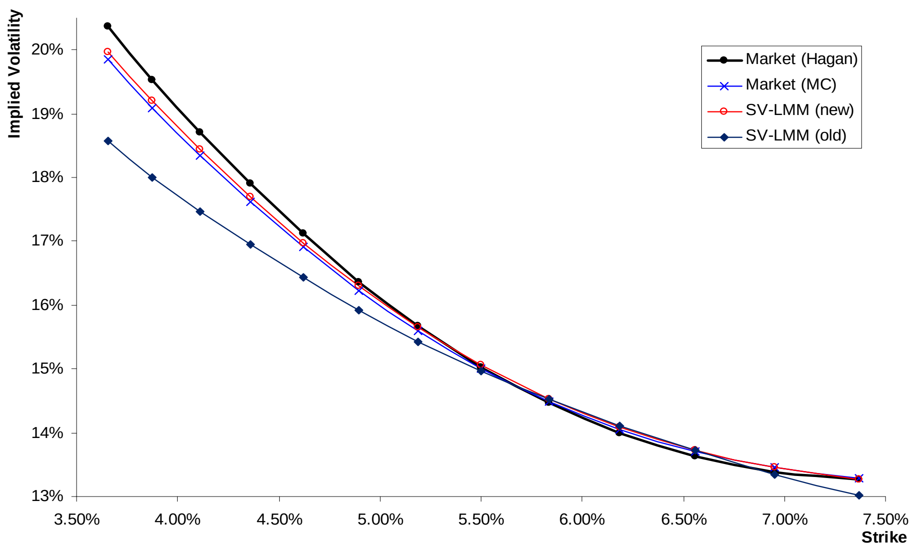

Figure 2: The accuracy of the new approximation for caplets for a 5-year caplet. The (market) SABR parameters were: $f(0) = 5.19\%$, $\sigma(0) = 3.5\%$, $\beta = 0.5$, $\phi = -0.45$ and $\nu = 33.5\%$.

3 From Caplets to Swaptions: Specification of the Correlation Structure

The prices of caplets only depend on the correlation among the forward rates and the volatilities. For products with more complex payoffs (e.g., swaptions), however, we must fully specify the parameters in Equations (7) to (12), i.e., we must also define the correlations among the forward rates, among the volatilities and the non-diagonal elements $\phi_{ij}, i \neq j$, as only the diagonal of the sub-matrix

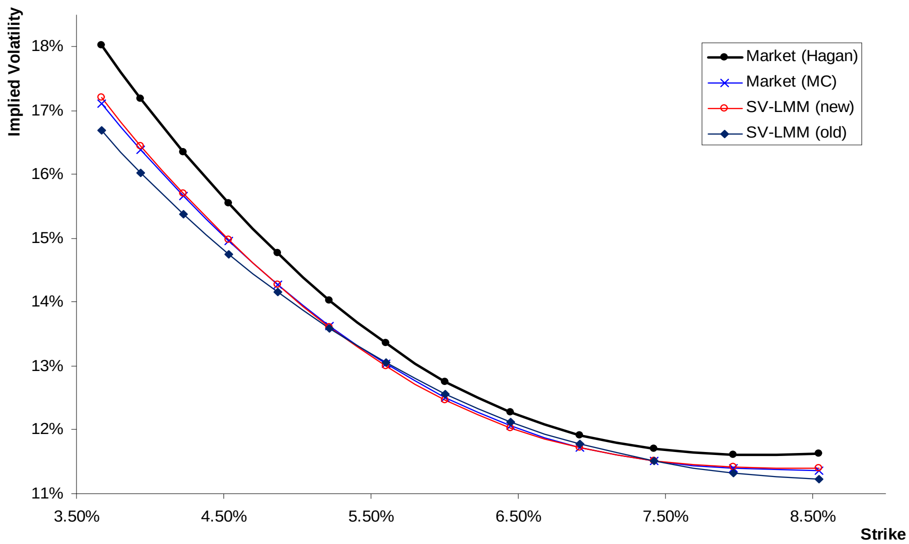

Figure 3: The accuracy of the new approximation for caplets for a 10-year caplet. The (market) SABR parameters were: $f(0) = 5.60\%$, $\sigma(0) = 3.05\%$, $\beta = 0.5$, $\phi = -0.395$ and $\nu = 28.25\%$.

$\boldsymbol{\phi}$ is available from the SABR caplet-related information. Let's therefore define the full correlation matrix as

$$\mathbf{P} = \begin{bmatrix} \boldsymbol{\rho} & \boldsymbol{\phi} \\ \boldsymbol{\phi}^{\dagger} & \boldsymbol{\theta} \end{bmatrix} \tag{35}$$

We parametrize the remaining blocks of the matrix as

$$\rho_{ij} = \eta_1 + (1 - \eta_1) \exp\left[-\lambda_1 \left(|T_i - T_j|\right)\right] \tag{36}$$

$$\theta_{ij} = \eta_2 + (1 - \eta_2) \exp\left[-\lambda_2 \left(|T_i - T_j|\right)\right] \tag{37}$$

$$\phi_{ij} = \mathrm{sign}(\phi_{ii}) \sqrt{|\phi_{ii}\phi_{jj}|} \exp\left[-\lambda_3 \left(T_i - T_j\right)^+ - \lambda_3 \left(T_j - T_i\right)^+\right] \tag{38}$$

The functional form used for Equations (36) and (37) is well known in the literature (see, e.g., Rebonato (2002), Brigo and Mercurio (2001)); it allows for an exponential decay of the correlation as a function of the distance between the two forward rates to a non-zero asymptotic value, $\eta_i, i = 1, 2$, with a decay constant $\lambda_i, i = 1, 2$. The main advantages of this specification is its parsimony, and analytical and numerical tractability (as calendar time t does not explicitly appear). The main drawback is that the decorrelation between variables separated by the same 'distance' in maturity, $|T_i - T_j|$, does not depend on T_i and T_j. So, the decorrelation between, say, a two-year and a one-year forward rate is the same as the decorrelation between a twelve-year and a eleven-year forward rate. This is unpleasant. However, Joshi (2001) and Rebonato (2002) show that this functional form is not purely ad hoc, and can be taken as the first-order term of a wide class of more general specifications. Furthermore, Rebonato (2002) shows that the pricing impact of the precise form of the instantaneous correlation function on swaptions is rather limited once the overall level of the decorrelation is correctly captured.[4]

We stress that a good choice of the correlation matrix is important to study the congruence between the caplet and the swaption markets. For the more limited goal of this paper, however, the results do not depend on the particular parametrization chosen, as they are all expressed simply in terms of the elements ρ_{ij}, θ_{ij}, and ϕ_{ij}, however obtained.

Let $SR_t^{\alpha\beta}$ be the forward swap rate spanned by $\beta - \alpha$ forward rates f_t^{α} to $f_t^{\beta-1}$:

$$SR_t^{\alpha\beta} = \sum_{i=\alpha}^{\beta-1} w_i f_t^i \tag{39}$$

We will often drop in the following the $\alpha\beta$ superscript and the summation limits unless they are required for clarity. We assume for SR_t SABR dynamics of the type[5]:

$$dSR_t = (SR_t)^B \, \Sigma_t dZ_t \tag{40}$$

$$\frac{d\Sigma_t}{\Sigma_t} = V dZ_t' \tag{41}$$

$$E[dZ_t dZ_t'] = \Phi dt \tag{42}$$

We also assume that the forward rates follow a LMM-SABR dynamics as in Section 2. As anticipated in the introductory section, for a given set of forward-rate parameters our goal is to approximate analytically (i.e., without making use of a Monte Carlo approximation) the SABR parameters of the swaption-based prices implied by the LMM-SABR dynamics of the forward rates. Therefore we need to 'guess' the initial values of the swap-rate volatility of volatility, Σ_0, the correlation between the volatility of the swap rate and the swap rate itself, Φ, the exponent B, and the volatility of the swap-rate volatility, V, as a function of the forward-rate parameters or functions, $g(\tau_i)$, β_i, $\mathbf{P}$, $h(\tau_i)$, f_0^i, $i = \alpha$ to $\beta - 1$.

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[4] We note in passing that, under the specifications above, the super-matrix $\mathbf{P}$ need not be positive definite for all choices of the parameters $\lambda_i, i = 1, 2, .., 4$. In our calculations we ensure that this is always the case.

[5] To help the reader through the maze of symbols, we have used the convention of employing upper-case symbols for swaptions and lower-case for forward rates. Whenever possible (sometimes across alphabets), the same letters have been used in lower or upper case for the same quantities relating to forward rates or swap rates.

4 Initial Approximations

Let's apply Ito's lemma to Equation (39) using the approximation[6] $\frac{\partial w_k}{\partial f_t^j} = 0$ and focus on the volatility part of the resulting expression:

$$\Sigma_t \left(SR_t\right)^B dZ_t = \sum_k \frac{\partial SR_t}{\partial f_t^k} \left(f_t^k\right)^{\beta_k} s_t^{T_k} dz_t^k$$

$$\simeq \sum_k w_k \left(f_t^k\right)^{\beta_k} s_t^{T_k} dz_t^k \tag{43}$$

Consider now the expressions

$$(\Sigma_t)^2 \left(SR_t\right)^{2B} = \sum_{k,m} w_k w_m \left(f_t^k\right)^{\beta_k} \left(f_t^m\right)^{\beta_m} s_t^{T_k} s_t^{T_m} \rho_{k,m} \tag{44}$$

and

$$\Sigma_t = \sqrt{\frac{\sum_{k,m} w_k w_m \left(f_t^k\right)^{\beta_k} \left(f_t^m\right)^{\beta_m} s_t^{T_k} s_t^{T_m} \rho_{k,m}}{\left(\sum_k w_k f_t^k\right)^{2B}}} \tag{45}$$

From Equation (44) we have:

$$(\Sigma_t)^2 = \frac{\sum_{k,m} w_k w_m \left(f_t^k\right)^{\beta_k} \left(f_t^m\right)^{\beta_m} s_t^{T_k} s_t^{T_m} \rho_{k,m}}{\left(SR_t\right)^{2B}} =$$

$$= \sum_{k,m=1,n_j} \left\{ w_k \frac{\left(f_t^k\right)^{\beta_k}}{\left(SR_t\right)^B} \right\} \left\{ w_m \frac{\left(f_t^m\right)^{\beta_m}}{\left(SR_t\right)^B} \right\} s_t^{T_k} s_t^{T_m} \rho_{k,m} \tag{46}$$

Define

$$W_k^t = w_k \frac{\left(f_t^k\right)^{\beta_k}}{\left(SR_t\right)^B} \tag{47}$$

Equation (46) becomes

$$\Sigma_t = \sqrt{\sum_{k,m=1,n_j} W_k^t W_m^t s_t^{T_k} s_t^{T_m} \rho_{k,m}} \tag{48}$$

Comparing with Equation (40) gives:

$$dSR_t = \left(SR_t\right)^B \sqrt{\sum_{k,m=1,n_j} W_k^t W_m^t s_t^{T_k} s_t^{T_m} \rho_{k,m}} dW_t^j \tag{49}$$

Up to this point no approximations have been made (apart from the avoid-

[6] This approximation is not required, and could be replaced by the more cumbersome expression in Jaeckel and Rebonato (2003). The effect is generally small, and exactly zero for a flat term structure of rates.

able $\frac{\partial w_k}{\partial f_t^j} = 0$) Let us now assume that the variation over time of the ratio $\left(f_t^k\right)^{\beta_k} / \left(SR_t\right)^B$ can be considered small compared with the variation in the functions $s_t^{T_m}$. This is plausible because swap rates are strongly correlated with the underlying forward rates. Then, following usual practice (see, e.g., Hull and White (2000), Jaeckel and Rebonato (2003)) , it is reasonable to assume that the 'weights' W_k^t could be frozen with little loss of precision. (See, however, Labordere (2006) for a discussion of the limitations of this approach.) This would give

$$\Sigma_t = \sqrt{\sum_{k,m=1,n_j} W_k^0 W_m^0 s_t^{T_k} s_t^{T_m} \rho_{k,m}} \tag{50}$$

We use this approximation as our starting point and we derive below further approximations for the quantities Σ_0 (initial volatility), Φ (swap-rate/swap-rate volatility correlation), V (volatility of volatility) and B (exponent) implied by forward-rate LMM-SABR parameters. We find that several, a priori equally plausible, approximations could be made. Interestingly enough, however, we find that surprisingly few provide the exact solution in the case of a one-period swaption (i.e., a caplet). We therefore use this condition as a powerful guide to devising our approximations.

5 Approximating the Initial Value of the Swap Rate, Σ_0 (First Route)

To approximate the value of Σ_0 we proceed as follows. After 'freezing the initial values', one can assume

$$(\Sigma_0)^2 T \simeq \int_0^T (\Sigma_t)^2 dt =$$

$$\sum_{k,m_j} W_k^0 W_m^0 \int_0^T s_t^{T_k} s_t^{T_m} \rho_{k,m} dt \tag{51}$$

$$\Sigma_0 = \sqrt{\frac{1}{T} \sum_{k,m} W_k^0 W_m^0 \int_0^T s_t^{T_k} s_t^{T_m} \rho_{k,m} dt} \tag{52}$$

where T is the expiry of the swaption, and T_i is the expiry of a particular forward rate (of course $T_i \geq T$). Further freezing the terms $k_t^{T_i}$ to their initial values gives the following approximation for the initial value of Σ_t, Σ_0:

$$\Sigma_0 = \sqrt{\frac{1}{T} \sum_{k,m} W_k^0 W_m^0 k_0^{T_k} k_0^{T_m} \int_0^{T_j} g_t^{T_k} g_t^{T_m} \rho_{k,m} dt} \tag{53}$$

One can verify that in the limit of a one-period swaption (caplet) Equation (53) gives

$$\Sigma_0 \equiv \sigma_0 = k_0^T \sqrt{\frac{1}{T}\int_0^T \left(g_t^T\right)^2 dt} \tag{54}$$

This expression coincides with Equation (18) and is therefore perfectly consistent with the caplet set-up.

6 Approximating Σ_0 (Second Route) and the Volatility of Volatility of the Swap Rate, V

Recall that the SABR dynamics of the swap rate are

$$dS = \Sigma S^B dZ \tag{55}$$

$$d\Sigma = \Sigma V dW \tag{56}$$

with $< dZdW >= \rho_{\text{SABR}}dt$. We now use the approximation obtained above,

$$(\Sigma_t)^2 \approx \sum_{i,j} w_i w_j g_t^i g_t^j k_t^i k_t^j \left(\frac{\left(f_t^i\right)^\beta}{S_t^B}\right)\left(\frac{\left(f_t^j\right)^\beta}{S_t^B}\right)\rho_{ij} \tag{57}$$

and we proceed as for caplets to obtain (dropping t subscript)

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \Sigma^2 dt\right] &= \mathbb{E}\left[\int_0^T \left(\sum_{i,j} w_i w_j g^i g^j k^i k^j \left(\frac{\left(f^i\right)^\beta}{S^B}\right)\left(\frac{\left(f^j\right)^\beta}{S^B}\right)\rho_{ij}\right)dt\right] \\
&= \sum_{i,j} w_i w_j \left(\int_0^T g^i g^j \mathbb{E}\left[k^i k^j \left(\frac{\left(f^i\right)^\beta}{S^B}\right)\left(\frac{\left(f^j\right)^\beta}{S^B}\right)\right]\rho_{ij} dt\right)
\end{aligned} \tag{58}$$

Again setting $w_i\left(\frac{\left(f^i\right)^\beta}{S^B}\right) = W_i$ and 'freezing', we have

$$\begin{aligned}
\left(\frac{\Sigma_0}{V}\right)^2\left(e^{V^2T}-1\right) &= \sum_{i,j}\left(\rho_{ij}\int_0^T g^i g^j \mathbb{E}\left[W_i W_j k^i k^j\right] dt\right) \\
&\approx \sum_{i,j}\left(\rho_{ij} W_i^0 W_j^0 \int_0^T g^i g^j \mathbb{E}\left[k^i k^j\right] dt\right)
\end{aligned} \tag{59}$$

We define the cross root-mean-square of the volatility of volatility by

$$\widehat{h}_{ij}(t) = \sqrt{\frac{1}{t}\int_0^t h^i(s)h^j(s)ds} \tag{60}$$

With this definition it is easy to show that

$$\mathbb{E}\left[k^i(t)k^j(t)\right] = k_0^i k_0^j \exp(\theta_{ij}\widehat{h}_{ij}(t)^2 t) \tag{61}$$

From this we have the expressions

$$\left(\frac{\Sigma_0}{V}\right)^2 \left(e^{V^2 T} - 1\right) = \sum_{i,j} \left(\rho_{ij} W_i^0 W_j^0 k_0^i k_0^j \int_0^T g^i g^j \exp(\theta_{ij}\widehat{h}_{ij}(t)^2 t) dt\right) \tag{62}$$

Taylor expanding both sides and equating terms of the same order finally gives

$$\Sigma_0 = \sqrt{\frac{1}{T}\sum_{i,j}\left(\rho_{ij} W_i^0 W_j^0 k_0^i k_0^j \int_0^T g^i g^j dt\right)} \tag{63}$$

$$V = \frac{1}{\Sigma_0 T}\sqrt{2\sum_{i,j}\left(\rho_{ij}\theta_{ij} W_i^0 W_j^0 k_0^i k_0^j \int_0^T g^i g^j \widehat{h}_{ij}(t)^2 t dt\right)} \tag{64}$$

Using this procedure, not only do we obtain an expression for V, but we also note that the equation we have obtained for the initial value of the swap rate, Σ_0, coincides with the expression we had obtained above using a different route. Finally, it is immediate to verify that Equation (64) coincides with Equation (32) in the one-period-swaption case.

7 Approximating the Swap-Rate/Swap-Rate -Volatility Correlation, Φ

In order to approximate the correlation between a swap rate and its volatility it would be tempting to suggest a simplistic expression such as

$$\Phi = \sum_{k=1,n_j} w_k \rho_k \tag{65}$$

This would however be inaccurate, the more so the more the forward rates are imperfectly correlated with each other. To see this, consider the limiting case when the volatilities of forward rates are strongly correlated with their own forward rates, but the forward rates themselves have zero correlation among themselves. In this situation, the correlation between the change in a given underlying forward rate and the swap rate could be low (the more so, the longer the swap), and so would therefore be the correlation between the swap rate and its volatility. This would however not be captured by Equation (65). We must ensure that our approximation reflects this intuition.

We want to choose the parameters of the forward-rate process in such a way that

$$E[dZ_t dZ_t'] = \Phi dt \tag{66}$$

Differentiating expression (50) gives:

$$d\Sigma_t = d\left(\sqrt{\sum_{k,m} W_k^0 W_m^0 s_t^{T_k} s_t^{T_m} \rho_{k,m}}\right) \tag{67}$$

Using Ito's lemma gives (see Appendix A):

$$\frac{d\Sigma_t}{\Sigma_t} = Adt + \sum_{k,m_j} W_k^0 W_m^0 \rho_{k,m} \Psi_t^k \Psi_t^m h_t^{T_k} dw_t^k$$

with

$$\Psi_t^k \equiv \frac{s_t^{T_k}}{\Sigma_t} \tag{68}$$

The terms in dt (also given in Appendix A) are irrelevant for the future discussion. So

$$VAR\left[\frac{d\Sigma_t}{\Sigma}\right] = \sum_{k,m,n,l=1,n_j} \left\{W_k^0 W_m^0 W_n^0 W_l^0 \Psi_t^k \Psi_t^m \Psi_t^n \Psi_t^l \rho_{k,m} \rho_{n,l} h_t^{T_k} h_t^{T_n} \theta_{kn}\right\} \tag{69}$$

Freezing the Ψ_t^k terms gives

$$V \approx \sqrt{\frac{1}{T_j} \sum_{k,m,n,l} \left\{W_k^0 W_m^0 W_n^0 W_l^0 \Psi_0^k \Psi_0^m \Psi_0^n \Psi_0^l \rho_{k,m} \rho_{n,l} \theta_{kn} \int_0^T h_t^{T_k} h_t^{T_n} dt\right\}} \tag{70}$$

From Equation (70) it would be tempting to try to achieve $E[dZ_t dZ_t'] = \Phi dt$ simply by approximating

$$\mathbb{E}\left[\frac{dSR}{SR^B}\frac{d\Sigma}{\Sigma}\right] =$$

$$= \mathbb{E}\left[\left(\sum_r W_r^0 k_0^{T_r} g_t^{T_r} dz_t^r\right)\left(\sum_{k,m} W_k^0 W_m^0 \rho_{k,m} \Psi_t^k \Psi_t^m h_t^{T_k} dw_t^k\right)\right] =$$

$$= \sum_{k,m,r} k_0^{T_r} W_r^0 W_k^0 W_m^0 \rho_{k,m} \Psi_t^k \Psi_t^m \phi_{rk} g_t^{T_r} h_t^{T_k} dt \tag{71}$$

i.e., one could set

$$\Phi = \frac{\sum_{k,m,r} k_0^{T_r} W_r^0 W_k^0 W_m^0 \Psi_t^k \Psi_t^m \rho_{k,m} \phi_{rk} \frac{1}{T} \int_0^T g_t^{T_r} h_t^{T_k} dt}{\Sigma_0 V} \tag{72}$$

or, after pulling the functions $g()$ inside the integral,

$$\Phi = \frac{\sum_{k,m,r} k_0^{T_k} k_0^{T_m} k_0^{T_r} W_r^0 W_k^0 W_m^0 \rho_{k,m} \phi_{rk} \int_0^T g_t^{T_k} g_t^{T_m} g_t^{T_r} h_t^{T_k} dt}{(\Sigma_0)^3 VT} \tag{73}$$

However, from Equations (73) or (72) together with eqn. (64) one can see that the correct one-period-swaption (caplet) limit will in general not be recovered unless

$$\frac{\int g_t^T h_t^T dt}{\left(2\int_0^T g(t)^2 \widehat{h_t}^2 t dt\right)^{1/2}} = 1 \tag{74}$$

or

$$\frac{\int_0^T \left(g_t^T\right)^3 h_t^T dt}{\left(\frac{1}{T}\int_0^T g(t)^2 dt\right)\left(2\int_0^T g(t)^2 \widehat{h_t}^2 t dt\right)^{1/2}} = 1 \tag{75}$$

according to whether approximation (72) or (73) are used. This will not happen for non-trivial cases. In order to fix this problem, we use Equations (72) and (73) as an indication of the overall structure of the expression we are looking for, and Equations (74) to (75) as a hint of the problems we must avoid. We therefore proceed by defining the matrix $\mathbf{\Omega}$ as

$$\Omega_{ij} = \frac{2\rho_{ij}\phi_{ij} W_i^0 W_j^0 k_0^i k_0^j \int_0^T g^i g^j \widehat{h}(t)^2 t dt}{\left(V\Sigma_0 T\right)^2} \tag{76}$$

From Equation 64 we then have

$$\Omega_{ij} \geq 0 \text{ and } \sum_{i,j} \Omega_{ij} = 1 \tag{77}$$

i.e. the quantities Ω_{ij} have the properties of weights. If we now propose the following expression for Φ

$$\Phi = \sum_{i,j} \Omega_{ij}\phi_{ij} \tag{78}$$

the structure of the approximation reflects the intuition mentioned at the start of the section and what suggested by Equations (72) and (73), but, at the same time, the correct one-period-swaption limit is recovered.

8 Approximating the Swap Rate Exponent, B

Finally, for the exponent B we simply set

$$B = \sum_{k=1,n_j} w_k \beta_k \tag{79}$$

This *Ansatz* is clearly heuristic, as the (approximate) sum of CEV-variables with exponent β is in general not a CEV-variable with the same exponent. We know however that in the log-normal case the approximation is good (see Rebonato (1998)), and that in the normal case is exact. As the CEV exponent β is between 1 and 0, the approximation should be at least a good as in the log-normal case, and increasingly better as β approaches 0.

It is trivial to verify that in the limit of a one-period swaption the approximation is exact.

9 Results

9.1 Comparison between Approximated and Simulation Prices

The most direct way (but not the only way - see below) to check the accuracy of the approximations above is, of course, to compare the swaption prices as obtained by a full Monte Carlo simulation with the prices produces by the approximations suggested above. We performed these checks by using the following inputs. For the forward-rate parameters we used values similar to those typically observed in the calibration to market caplet prices of the forward-rate-based LMM-SABR model. As for the various components of the correlation super-matrix $\mathbf{P}$, we made the following choices:

- in the absence of direct market information we used as inputs a variety of correlation structures for the volatility/volatility correlations, $\boldsymbol{\theta}$;
- the diagonal of the sub-matrix of correlations between the forward rates and their volatilities, $\boldsymbol{\phi}$, were obtained from the specified forward-rate LMM-SABR model;
- the off-diagonal elements of the sub-matrix $\boldsymbol{\phi}$ were obtained using formula (38);
- finally, the forward-rate/forward-rate correlations were set around typical values reported in the literature (see Rebonato (2002) and references therein).

For each set of forward-rate parameters and correlations we calculated the 'true' swaption prices by running a Monte Carlo simulation of the forward-rate processes. We then used the expressions presented above to calculate the swaption SABR parameters. The swaption prices from this simulation were then compared with the prices obtained using the approximate SABR swaption parameters Σ_0^j, Φ_i, B_j, V_j. In order to facilitate comparison, we input these parameters both in a separate Monte Carlo simulation of the true SABR process and in the Hagan (2002) SABR formula.

The accuracy of the approximations above is shown in Figures 4 to 10. To put these numbers in context in the same figures we also display the accuracy of the SABR asymptotic approximation. So,

- the curves labelled 'SV-LMM MC' report the implied volatilities corresponding to the swaption prices obtained running a Monte Carlo simulation of the LMM-SABR forward-rate processes;
- the curves labelled 'New Approx (MC)' report the implied volatilities corresponding to the swaption prices obtained running a Monte Carlo simulation of the SABR swaption-rate process with the parameters 'guessed' using our approximation;

- the curves labelled 'New Approx (Hagan)' report the implied volatilities corresponding to the swaption prices obtained inputting the parameters 'guessed' using our approximation in the Hagan (2002) SABR formula;
- the curves labelled 'Market (SABR)' report the implied volatilities corresponding to the swaption prices obtained inputting in the Hagan formula the parameters used by the market for swaptions on the same day the forward-rate parameters were obtained;
- the curves labelled 'SABR MC' report the implied volatilities corresponding to the swaption prices obtained running a Monte Carlo simulation of the SABR swaption-rate process with the parameters used by the market for swaptions on the same day the forward-rate parameters were obtained.

We stress that no direct comparison should be made between the curves labelled 'New Approx' and the curves labelled 'Market (SABR)' and 'SABR MC'. The two distinct sets of curves are presented together only to give a yardstick to gauge the accuracy of the new approximation.

From these graphs we see that in the worst cases our approximation is of the same order of accuracy as the asymptotic expansion of the SABR model, and often much better. We note that the approximations display greater accuracy for 'long-tail' swaptions than for 'short tail' ones and for short-expiry than for long-expiry swaptions For instance, the 10x10 curves for the swaption implied volatilities obtained using our approximate parameters in a MC simulation and via direct simulation of the forward-rate LMM-SABR process are virtually on top of each other: that is why in Fig 10 there appears to be only four curves. On the other hand, the accuracy of the approximation is worst for the 5x2 swaption. Since the caplet approximation is excellent (see Figs 2 and 3), this suggests that one reason of weakness of the approximations we have proposed stems from the difficulty in capturing correctly the integral cross-terms between the functions $h()$ and $g()$ when these change rapidly. This would explain why 'long-tail' swaptions are well captured: the longer the underlying swap rate, the larger the number of forward rates that, by swaption expiry, is still in a very flat-volatility environment, the smaller the importance of the correct handling of the time variation of the functions $g()$ and $h()$. Similarly, if the expiry is long, for most of the time the cross-integrals of the functions $h()$ and $g()$ will contain rather flat curves.

9.2 Comparison between Parameters from the Approximations and the Simulations.

An alternative way to understand the relative accuracy of our approximations is the following. We first calculated using a Monte Carlo simulation the prices of the swaptions for several strikes implied by a given set of forward-rate LMM-SABR parameters. As a second step, we fitted a swaption-based SABR 'model' (i.e., the Hagan (2002) formula) to these prices. To avoid multiple minima - in the SABR model the exponent and the correlation coefficient tend to 'play

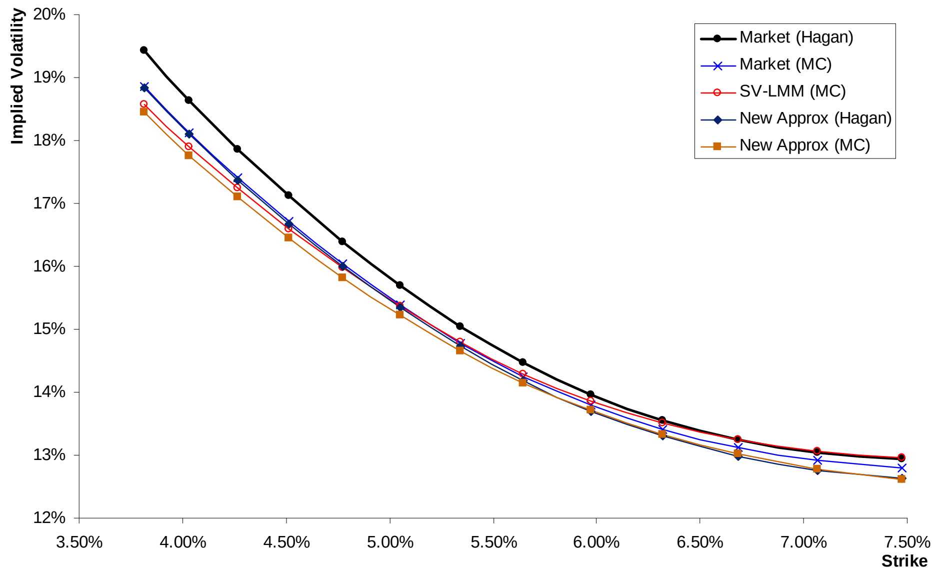

Figure 4: 5Y into 2Y Swaption

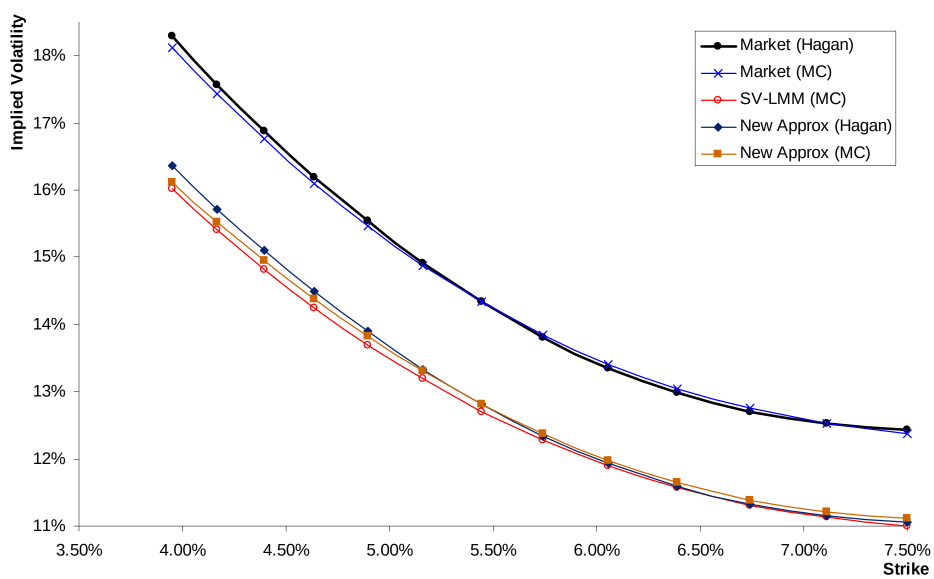

Figure 5: 5Y into 5Y Swaption

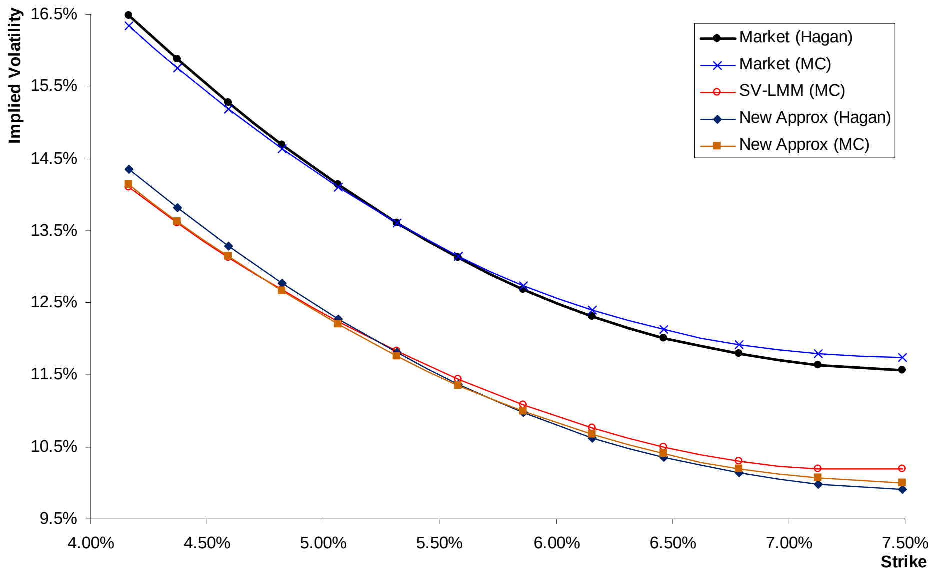

Figure 6: 5Y into 10Y Swaption

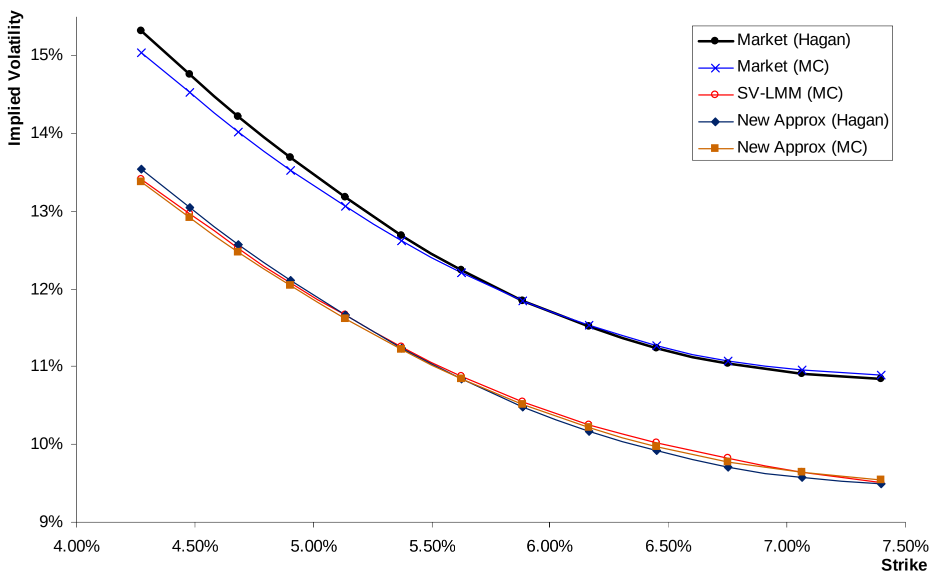

Figure 7: 5Y into 15Y Swaption

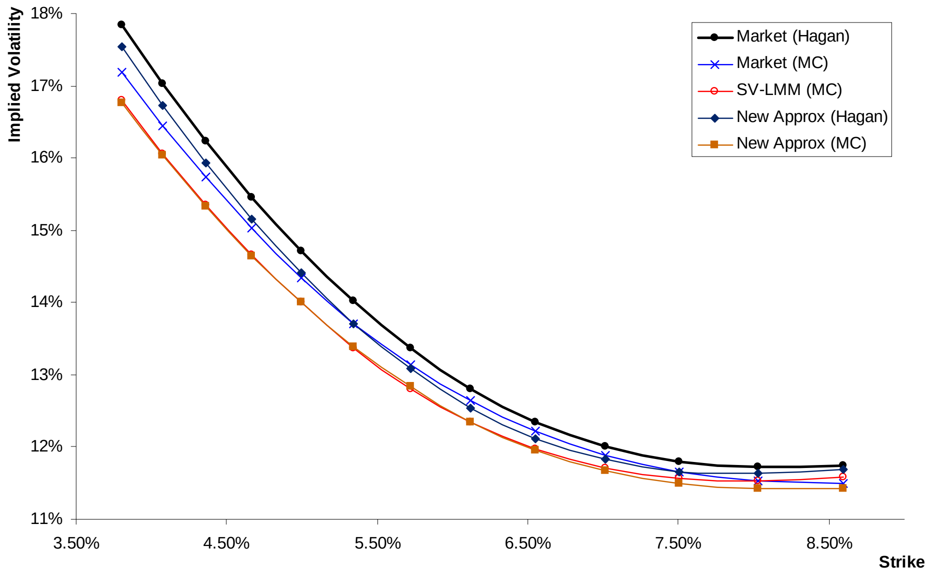

Figure 8: 10Y into 2Y Swaption

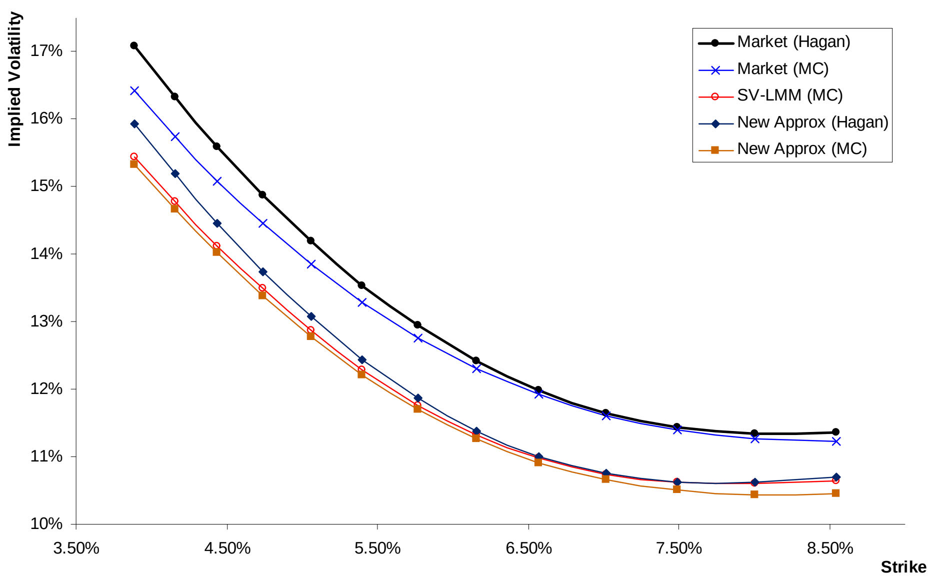

Figure 9: 10Y into 5Y Swaption

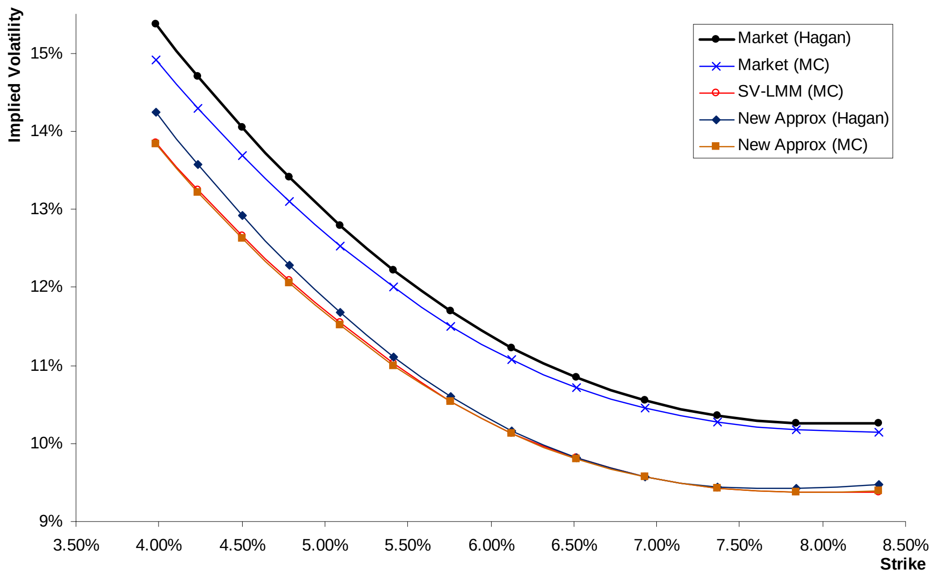

Figure 10: 10Y into 10Y Swaption

against each other' (see, e.g., Castagna Mercurio and Tarenghi (2007)) - we anchored the numerical search around the exponent given by Equation (79). We then compared the SABR coefficients obtained by this search with the coefficients estimated using the approximations above. This analysis allows us to appreciate which of the various approximations are more effective.

The results are shown in table 1 below. For each section, the row labelled 'MC LMM' displays the SABR coefficients fitted to the prices obtained using the Monte Carlo simulation of the forward-rate processes, and the row labelled 'Approx' displays the SABR coefficients as guessed by our approximate formulae[7] using as input the parameters of the forward-rate processes and the matrix **P**.

We see that the approximation (53) estimates in all conditions very accurately the initial value of the volatility, Σ_0 (minimum relative error 0.2%, maximum relative error 0.8%, average relative error 0.5%). As this parameter is mainly linked to the level of the smile, we understand why our approximate curves are always well centred at the at-the-money level. The approximation for the parameter V, that mainly controls the degree of curvature of the smile, is the second most accurate (minimum relative error 0.5%, maximum relative error 4.9%, average relative error 2.4%). The approximation for the parameter

[7] Actually it shows the SABR fit to the smile produced by running a Monte Carlo on the SABR dynamics 'guessed' by our approximation. This is more appropriate when comparing to the LMM.

		Σ_0	B	Φ	V
5Y Caplet	SV-LMM	3.51%	0.5	-43.7%	30.46%
(Fig. 2)	Approx	3.49%	0.5	-42.7%	30.45%
10Y Caplet	SV-LMM	2.98%	0.5	-36.41%	25.85%
(Fig. 3)	Approx	3.00%	0.5	-36.73%	24.87%
5 X 2 Swaption	SV-LMM	3.35%	0.5	-37.38%	29.91%
(Fig. 4	Approx	3.34%	0.5	-40.88%	28.51%
5 X 5 Swaption	SV-LMM	2.92%	0.5	-40.44%	26.92%
(Fig. 5)	Approx	2.95%	0.5	-40.58%	27.08%
5 X 10 Swaption	SV-LMM	2.65%	0.5	-35.68%	25.80%
(Fig. 6)	Approx	2.64%	0.5	-38.25%	26.08%
5 X 15 Swaption	SV-LMM	2.55%	0.5	-40.18%	24.98%
(Fig. 7)	Approx	2.54%	0.5	-39.22%	25.52%
10 X 2 Swaption	SV-LMM	2.84%	0.5	-33.22%	26.52%
(Fig. 8)	Approx	2.86%	0.5	-35.13%	25.56%
10 X 5 Swaption	SV-LMM	2.62%	0.5	-34.12%	25.55%
Fig. 9)	Approx	2.61%	0.5	-35.58%	24.65%
10 X 10 Swaption	SV-LMM	2.46%	0.5	-36.17%	24.26%
(Fig. 10)	Approx	2.45%	0.5	-35.59%	24.38%

Table 1: Fitted SABR parameters for full LMM Monte Carlo and our approximation

Φ, that mainly controls the slope of the smile, is the least accurate (minimum relative error 0.3%, maximum relative error 8.6%, average relative error 4.2%). This explains why the small errors in implied volatilities tend to have opposite signs moving from deeply in- to deeply out-of-the-money.

10 Conclusions and Suggestions for Future Work

We have presented analytical approximation of the swaption prices implied by a set of parameters for the forward-rate-based SABR-LMM model. In the range of parameter values associated with market fits, the approximations have been shown to be very accurate and to do at least as well as, and very often better than, the generally-market-accepted SABR formula (stemming from the approximate nature of the underlying asymptotic expansion).

En route to obtaining these results we have presented an improvement on the approach presented in Rebonato (2007) to calibrating the same forward-rate-based SABR-LMM model to caplet prices.

This work can be of great use in calibrating a forward-rate-based SABR-LMM to (subsets of) the swaption matrix, or in obtaining a joint calibration to caplets and swaptions for mixed products (e.g., products with LIBOR-based payoffs and callability features).

The approximations can also be useful in order to obtain a consistent drift correction for CMS products in a forward-rate-based LMM approach. Hagan (2003) has in fact shown that the correction can be expressed as an integral over swaption prices. From a set of forward-rate parameters and correlations, in fact, one can use our approximations to obtain directly the SABR parameters for swaptions, obtain from these the swaption Black 'implied volatilities' using Hagan (2002) formula and input these into the Black formula for swaptions under the sign of the integral as in Hagan (2003).

Finally, the approximations we have presented would be very useful to investigate the congruence of the swaption and caplet markets and to explore whether the correlation super-matrix $\mathbf{P}$ can be profitably 'implied' from market prices of caplets and swaptions.

A Original Approximation of Φ

Start from

$$\frac{d\Sigma_t}{\Sigma_t} = \frac{\sum_{k,m=1,n_j} W_k^0 W_m^0 \rho_{k,m} d(s_t^{T_k} s_t^{T_m})}{2\left(\Sigma_t\right)^2} \tag{80}$$

Consider the terms $d(s_t^{T_k} s_t^{T_m})$. Each term is equal to

$$d(s_t^{T_k} s_t^{T_m}) = \left(s_t^{T_m} ds_t^{T_k} + s_t^{T_k} ds_t^{T_m} + ds_t^{T_m} ds_t^{T_k}\right) \tag{81}$$

Taking one term at a time gives:

$$\begin{aligned} s_t^{T_m} ds_t^{T_k} &= k_t^{T_m} g_t^{T_m} d\left(k_t^{T_k} g_t^{T_k}\right) = \\ & k_t^{T_m} g_t^{T_m}\left(g_t^{T_k} k_t^{T_k} h_t^{T_k} dw_t^k + k_t^{T_k} g_t^{\prime T_k} dt\right) \end{aligned} \tag{82}$$

and

$$\begin{aligned} ds_t^{T_m} ds_t^{T_k} &= d\left(k_t^{T_m} g_t^{T_m}\right) d\left(k_t^{T_k} g_t^{T_k}\right) = \\ &= g_t^{\tau_k} g_t^{\tau_m} k_t^{T_m} k_t^{T_k} h_t^{\tau_k} h_t^{\tau_m} \theta_{mk} dt \end{aligned} \tag{83}$$

Putting the various terms together gives:

$$\begin{aligned} d(s_t^{T_k} s_t^{T_m}) &= \left(s_t^{T_m} ds_t^{T_k} + s_t^{T_k} ds_t^{T_m} + ds_t^{T_m} ds_t^{T_k}\right) = \\ & k_t^{T_m} g_t^{T_m}\left(k_t^{Tk} g_t^{T_k} h_t^{T_k} dw_t^k + k_t^{T_k} g_t^{\prime T_k} dt\right) + \\ & k_t^{T_k} g_t^{T_k}\left(k_t^{T_m} g_t^{T_m} h_t^{T_m} dw_t^m + k_t^{T_m} g_t^{\prime T_m} dt\right) + \\ & k_t^{T_k} k_t^{T_m} g_t^{T_k} g_t^{T_m} h_t^{T_k} h_t^{T_m} \theta_{km} dt \end{aligned} \tag{84}$$

After collecting terms in dt and dw we have:

$$\begin{aligned} d(s_t^{T_k} s_t^{T_m}) \quad = \quad & \\ & \left[k_t^{T_m} k_t^{T_k}\left(g_t^{T_m} g_t^{\prime T_k} + g_t^{T_k} g_t^{\prime T_m}\right) + k_t^{T_k} k_t^{T_m} g_t^{T_k} g_t^{T_m} h_t^{T_k} h_t^{T_m} \theta_{km}\right] dt \\ & + k_t^{T_m} k_t^{T_k} g_t^{T_m} g_t^{T_k} h_t^{T_k} dw_t^k + k_t^{T_k} k_t^{T_m} g_t^{T_k} g_t^{T_m} h_t^{T_m} dw_t^m \end{aligned}$$

Equation (80) therefore becomes

$$\begin{aligned} \frac{d\Sigma_t}{\Sigma_t} &= \frac{\sum_{k,m} W_k^0 W_m^0 \rho_{k,m} d(s_t^{T_k} s_t^{T_m})}{2\left(\Sigma_t\right)^2} \\ &= \frac{1}{2\left(\Sigma_t\right)^2} \sum_{k,m} \left\{W_k^0 W_m^0 \rho_{k,m}\left[k_t^{T_m} k_t^{T_k}\left(g_t^{T_m} g_t^{\prime T_k} + g_t^{T_k} g_t^{\prime T_m}\right) + k_t^{T_k} k_t^{T_m} g_t^{T_k} g_t^{T_m} h_t^{T_k} h_t^{T_m} \theta_{km}\right]\right\} dt \\ & + \sum_{k,m}\left\{\frac{W_k^0 W_m^0 \rho_{k,m} k_t^{T_k} k_t^{T_m} g_t^{T_m} g_t^{T_k} h_t^{T_k}}{\left(\Sigma_t^j\right)^2} dw_t^k\right\} \end{aligned} \tag{85}$$

Call A all the terms in dt. Also, define

$$\Psi_t^k \equiv \frac{s_t^{T_k}}{\Sigma_t} \tag{86}$$

Then

$$\frac{d\Sigma_t}{\Sigma_t} = Adt + \sum_{k,m=1,n_j} W_k^0 W_m^0 \rho_{k,m} \Psi_t^k \Psi_t^m h_t^{T_k} dw_t^k \tag{87}$$

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